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Dreidel lasts $O(n^2)$ spins

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Abstract

We show that the expected number of spins in a game of dreidel is $O(n^2)$, where n is the number of tokens in the possession of each player at the beginning of the game. The implied constant depends on the number of players.

1. Introduction

Dreidel is a popular game played during the festival of Chanukkah. Players start with an equal number of tokens, and contribute one token each to a common pot. They then take turns spinning a four-sided top, called the dreidel. Depending on the side showing up, the spinner does one of the following:

Nisht (N): Nothing.
Ganz (G): Takes all the tokens in the pot.
Halb (H): Takes (the smaller) half of the number of tokens in the pot.
Shtel (S): Donates one token to the pot.

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Whenever the pot is empty, all the players *ante up*, i.e., donate one token each to the pot. Players lose, and *go home*, when they are required to donate a token to the pot, but cannot. The last survivor wins. The winner also goes home.

Feinerman [2] and Trachtenberg [6] investigated the fairness of a simplified model of dreidel. Zeilberger [7] conjectured that the expected number of spins in a game of dreidel between two players starting with n tokens each is $O(n^2)$. Later, Banderier (see [1]) conjectured that even in a multi-player game, the expected number of spins until first ruin is $O(n^2)$. We show that the expected duration of a game of dreidel where the players start with n tokens each is at most $c_k n^2$, where c_k is a constant that depends only on k. We begin by proving Zeilberger's conjecture using Markov chains, and then give an independent combinatorial proof of Banderier's conjecture.

2. The case $k = 2$

Throughout this section, we shall immerse dreidel in a Markov chain, and allow enough additional states. Explanation of terminology and proofs of standard results can be found in [3].

Let $\Lambda = 2n + 3$. We consider a Markov chain M on an infinite state space U, where each state is indexed by a triple (x, y, z) with x denoting the number of tokens in the pot, y denoting the number of tokens modulo Λ in the possession of P_1, and $z = i$ if and only if P_i plays next. Let $z^* \doteq 3 - z$.

The states reachable from $s_1 = (x_1, y_1, z_1)$ via a single transition are $\lambda_G(s_1) \doteq (2, y_1 + x_1 - 1, z_1^*)$, $\lambda_H(s_1) \doteq (\lceil \frac{x_1}{2} \rceil, y_1 + \lfloor \frac{x_1}{2} \rfloor, z_1^*)$, $\lambda_N(s_1) \doteq (x_1, y_1, z_1^*)$ and $\lambda_S(s_1) \doteq (x_1 + 1, y_1 - 1, z_1^*)$.

The initial state is $s_0 \doteq (2, n - 1, 1)$. The end-states are precisely the non-dreidel states. Moreover, U can be partitioned into disjoint subsets $A_k \doteq \{(x, y, z) \in U\colon\ x = k\}$.

Observe that the first coordinates of the state space form a Markov chain M_1. We observe that this chain is irreducible by considering

$$(x) \xrightarrow{\lambda_H \lambda_H \cdots \lambda_H} (1) \xrightarrow{\lambda_S \lambda_S \cdots \lambda_S} (y).$$

Since the set of timesteps on which any state can be reached is cofinite (consider $\lambda_N \lambda_N \cdots \lambda_N$), the chain is aperiodic. Finally, observe that the mean return time to the state (2) is at most E_g, the expected time for a Ganz. Since $E_g = \sum_{k=1}^{\infty} \frac{k}{4} (\frac{3}{4})^{k-1} = 4$, it follows that the mean return time to the state (2) is finite. Thus the chain is positive recurrent. It follows that M_1 is ergodic.

Let π_{ij} denote the transition probability from state (i) to state (j), and let π_j denote the stationary probability of M_1 being in state (j). Then, $\pi_j = \sum_{i=1}^{\infty} \pi_{ij} \pi_i$. From these equations, it can be easily shown that $\pi_2 \geqslant \frac{6}{13}$, an improvement over the calorie-free estimate $\pi_2 \geqslant \frac{1}{4}$.

To show that M is irreducible, consider an arbitrary pair of states $s_1 = (x_1, y_1, z_1)$, $s_2 = (x_2, y_2, z_2)$. Since $\lambda_N(x_1, y_1, z_1) = (x_1, y_1, z_1^*)$, we can assume that $z_1 = z_2 = 1$. Note that

$$(x_1, y_1, 1) \xrightarrow{\lambda_S \lambda_N \cdots \lambda_S \lambda_N} (x_1', y_2, 1) \xrightarrow{\lambda_N \lambda_H \cdots \lambda_N \lambda_H} (1, y_2, 1) \xrightarrow{\lambda_N \lambda_S \cdots \lambda_N \lambda_S} (x_2, y_2, 1).$$

Let $p_{ij}^{(k)}$ denote the transition probability from state i to state j in exactly k timesteps. Recall that the period of a state is the largest integer d such that $p_{ii}^{(k)} \neq 0 \Rightarrow d|k$. Since $p_{ii}^{(2)} \geqslant \frac{1}{16}$ (consider $\lambda_N \lambda_N$) and $p_{ii}^{(2k+1)} = 0$ (consider the third coordinate), it follows that $d = 2$. Thus all states have period 2.

Now consider a new Markov chain M' with state space consisting of the states $(x, y, 1)$ and transition probabilities given by $q_{ij} = p_{ij}^{(2)}$, where p_{ij} are the transition probabilities of M. It follows from the above arguments that M' is irreducible and aperiodic. Note that for any fixed i, $\sum_{j \in A_2} q_{ij}^{(k)} \geqslant \frac{1}{4}$ for all k. Since $|A_2|$ is finite, there exists $j \in A_2$, such that $\lim_{k\to\infty} q_{ij}^{(k)} > 0$. Thus there exist stationary probabilities π_j^*.

A sequence in M starting at the initial state is said to be *fast* if it reaches an end state before returning to the initial state, and is said to be *slow* if it returns to the initial state before reaching the end state. Let p_f (respectively p_s) denote the probability that a sequence starting at the initial state is a fast (respectively slow) sequence. Since the chain is positive recurrent, the sequence returns to the initial state with probability 1. Therefore, $p_f + p_s = 1$.

Let μ_0 denote the mean return time, i.e., the expected number of timesteps to return to the initial state. We have, $\mu_0 = p_f \mu_f + p_s \mu_s$, where μ_f and μ_s denote the mean return times for fast and slow sequences.

Observe that the definition of the second coordinate ensures that it is not possible to make an illegal move from a dreidel state to another dreidel state without passing through an end state. Therefore, a dreidel game ends without returning to the initial state with probability p_f, and returns to the initial state before ending with probability p_s. The former shall be called fast games and the latter, slow games.

Let μ_d denote the mean duration of a dreidel game, and let μ_{df} and μ_{ds} denote the mean duration of fast and slow dreidel games respectively. Note that $\mu_{ds} = \mu_s + \mu_d$ and $\mu_{df} \leqslant \mu_f$. Furthermore,

$$\mu_d = p_f \mu_{df} + p_s \mu_{ds} = p_f \mu_{df} + p_s(\mu_s + \mu_d).$$

It follows that

$$\mu_d = \frac{p_f \mu_{df} + p_s \mu_s}{1 - p_s} \leqslant \frac{p_f \mu_f + p_s \mu_s}{p_f} = \frac{\mu_0}{p_f}$$

Since $\pi_j^* = \pi_k^*$ for all $j, k \in A_2$ (by symmetry), we have $\pi_j^* = \frac{\pi_2}{\Lambda}$. Let μ_0' denote the mean return time for the initial state in M'. We have, $\mu_0' = \frac{1}{\pi_j^*} = \frac{\Lambda}{\pi_2}$. It follows that $\mu_0 = \frac{2\Lambda}{\pi_2} \leqslant \frac{13(2n+3)}{3}$.

We now derive a lower bound for p_f.

Let $P[y_1, z_1; y_2, z_2; y_3, z_3]$ denote the probability of reaching $(2, y_2, z_2)$ before $(2, y_3, z_3)$ given that we start at $(2, y_1, z_1)$. By an extension of notation, given a set of states S in M, $P[y_1, z_1; y_2, z_2; S]$ shall denote the probability of reaching $(2, y_2, z_2)$ before any of the states in S given that we start at $(2, y_1, z_1)$.

Let $\bar{a}$, $a \oplus b$ and $a \ominus b$ denote $-a$ mod Λ, $a + b$ mod Λ and $a - b$ mod Λ, respectively. The following identities are easily verified:

- *Duality*: $P[y_1, z_1; y_2, z_2; y_3, z_3] = P[\overline{y_1} \ominus 2, z_1^*; \overline{y_2} \ominus 2, z_2^*; \overline{y_3} \ominus 2, z_3^*]$
- *Complementarity*: $P[y_1, z_1; y_2, z_2; y_3, z_3] = 1 - P[y_1, z_1; y_3, z_3; y_2, z_2]$
- *Translation Invariance*: $P[y_1, z_1; y_2, z_2; y_3, z_3] = P[y_1 \oplus m, z_1; y_2 \oplus m, z_2; y_3 \oplus m, z_3]$.

Let $A_m^{y_1} = P[y_1, 1; y_1 \oplus m, 2; y_1 \ominus 1, 1]$. We have,

$$\begin{aligned}
\frac{A_{m+1}^{y_1}}{A_m^{y_1}} &\geqslant P\big[y_1 \oplus m, 2; y_1 \oplus (m+1), 2; y_1 \ominus 1, 1\big] \\
&= P\big[\overline{(y_1 \oplus m)} \ominus 2, 1; \big(\overline{y_1 \oplus (m+1)}\big) \ominus 2, 1; \big(\overline{y_1 \ominus 1}\big) \ominus 2, 2\big] \ (\textit{Duality}) \\
&= P\big[y_1, 1; y_1 \ominus 1, 1; y_1 \oplus (m+1), 2\big] \ (\textit{Translation Invariance}) \\
&= 1 - P\big[y_1, 1; y_1 \oplus (m+1), 2; y_1 \ominus 1, 1\big] \ (\textit{Complementarity}) \\
&= 1 - A_{m+1}^{y_1}.
\end{aligned}$$

Since $A_1^{y_1} \geqslant 1/4$ (consider λ_G), it follows from induction that $A_m^{y_1} \geqslant \frac{1}{m+3}$.

Let $B_m^{y_1} \doteq P[y_1, 1; y_1 \ominus m, 2; y_1 \oplus 1, 1]$. As before, it can be shown that $\frac{B_{m+1}^{y_1}}{B_m^{y_1}} \geqslant 1 - B_{m+1}^{y_1}$. Since $B_1^{y_1} \geqslant 1/64$ (consider $\lambda_S \lambda_H \lambda_N$), it follows from induction that $B_m^{y_1} \geqslant \frac{1}{m+63}$.

Let $S_1 = \{(2, n-1, 1), (2, n-2, 1)\}$ and $S_2 = \{(2, n-1, 1), (2, n, 1)\}$. Note that $\omega_1 \doteq P[n-1, 1; n, 1; S_1] \geqslant \frac{1}{8}$ (consider $\lambda_G \lambda_N$ and $\lambda_H \lambda_S$). Similarly, $\omega_2 \doteq P[n-1, 1; n-2, 1; S_2] \geqslant \frac{1}{8}$ (consider $\lambda_N \lambda_G$ and $\lambda_S \lambda_H$). Now,

$$\begin{aligned}
p_f &\geqslant P[n-1, 1; 2n+1, 2; n-1, 1] \\
&\geqslant \omega_1 P[n, 1; 2n+1, 2; n-1, 1] + \omega_2 P[n-2, 1; 2n+1, 2; n-1, 1] \\
&\geqslant \frac{1}{8} A_{n+1}^{n} + \frac{1}{8} B_n^{n-2} \geqslant \frac{1}{8(n+4)} + \frac{1}{8(n+63)} \geqslant \frac{1}{4(n+63)}.
\end{aligned}$$

Thus, $\mu_d \leqslant \frac{\mu_0}{p_f} \leqslant \frac{104n^2}{3} + o(n^2)$. This completes the proof. □

3. The general case

Let $P_1, P_2, \ldots, P_k$ denote the players, in the order in which they spin the dreidel. We introduce three variants of the game of dreidel.

Hyperdreidel works like dreidel, except that the players do not necessarily start with an equal number of tokens. *Slowdel* also works like dreidel, except that it is divided into *epochs*, and allows overdraft, so that the players can continue to play with a negative number of tokens. Define k spins to be a *round*. An epoch ends when the last spin in a round results in a Ganz (for player P_k). The ante up that follows is also part of the same epoch.

A player loses if and only if he or she has a negative number of tokens at the end of an epoch. *Hyperslowdel* is the slowdel analogue of hyperdreidel.

Clearly, the slowdel analogue of any instance of a game of dreidel or hyperdreidel lasts at least as many spins.

Consider a hyperslowdel game where P_k starts with W_0 tokens, $0 \leqslant W_0 \leqslant k(n-1)$. Let W_i denote the number of tokens P_k has at the end of the ith epoch. For $i \geqslant 1$, we define $Y_i = W_i - W_{i-1}$ to be the *payoff* of P_k during the ith epoch. Note that $\{Y_i\}$ is a set of independent and identically distributed random variables, with $S_m \doteq \sum_{i=1}^{m} Y_i = W_m - W_0$. Let $\mu \doteq E(Y_1)$ and $\sigma^2 \doteq \operatorname{Var}(Y_1) = E(Y_1^2) - \mu^2$. Let

$$T \doteq \inf_{j \in \mathbb{N}} \{ j\colon\ S_j < -W_0 \text{ or } S_j > k(n-1) - W_0 \}$$

so that P_k goes home at the end of the Tth epoch. Observe that T is a stopping time with respect to $\{Y_i\}$.

Lemma 1. $E(T) = O(n^2)$.

Proof. We will first show that $E(T)$ is finite. Define n epochs to be an *age*. Note that an epoch consisting of $k-1$ Shtels followed by a Ganz gives P_k a payoff of $2k-2$ units, and occurs with probability 4^{-k}. Thus the probability that all n epochs in a given age are of the above type is $\delta \doteq 4^{-kn}$. If we ever have such an age in a game, P_k clearly wins, and we say that P_k won by a *landslide*. Clearly, the expected number of ages before P_k wins by a landslide is given by $\sum_{j=1}^{\infty} j(1-\delta)^{j-1}\delta = 4^{kn}$. Thus the expected number of epochs in a game of hyperslowdel is at most $n4^{kn}$. Similarly, it can be shown that $E(T^2)$ is also finite.

Let $p_\omega(t)$ be the probability that the final epoch lasts at least t rounds. Note that any epoch can be turned into a final epoch by replacing the last round with a sequence of $kn-k-1$ Shtels followed by a Ganz. It follows that

$$p_\omega(n+\ell+1) \leqslant \frac{3^{n+\ell}}{3^{n+\ell} + 4^{\ell+k+n-kn}} < \frac{3^{n+\ell}}{4^{\ell+n-kn}} < \left(\frac{99}{100}\right)^{\ell} \quad \text{for } \ell \geqslant 5kn.$$

Let $s = kq + r,\ 0 \leqslant r < k$. Observe that if $|S_T| \geqslant 2kn + s$, then the last payoff Y_T must satisfy $|Y_T| \geqslant kn + k + s$, which is possible only if the final epoch lasts at least $kn + s + 1$ spins, since the number of tokens in the pot can go up only by one unit at a time. Therefore,

$$P\big(|S_T| \geqslant 2kn + s\big) \leqslant P\big(|Y_T| \geqslant kn + k + s\big) \leqslant p_\omega(n+q+1) < \left(\frac{99}{100}\right)^{q} \quad \text{for } s \geqslant 5k^2 n.$$

We have,

$$\begin{aligned} E\big(|S_T|\big) &= \sum_{i=1}^{\infty} P\big(|S_T| \geqslant i\big) \leqslant kn(5k+2) + \sum_{s > 5k^2 n} P\big(|S_T| \geqslant 2kn + s\big) \\ &\leqslant kn(5k+2) + k \sum_{q=5k}^{\infty} \left(\frac{99}{100}\right)^{q}. \end{aligned}$$

Similarly,

$$\begin{aligned} E\left(S_T^2\right) &= \sum_{i=1}^{\infty}(2i-1)P\left(|S_T| \geqslant i\right) \\ &\leqslant n^2\left(5k^2+2k\right)^2 + \sum_{s>5k^2n}(4kn+2s)P\left(|S_T| \geqslant 2kn+s\right) \\ &\leqslant n^2\left(5k^2+2k\right)^2 + 2k^2\sum_{q=5k}^{\infty}(2n+q+1)\left(\frac{99}{100}\right)^q. \end{aligned}$$

Therefore, $E(|S_T|) = O(n)$ and $E(S_T^2) = O(n^2)$.

Let $t-1 = ku+v$, $0 \leqslant v \leqslant k-1$. Note that $P(|Y_1| \geqslant t) \leqslant (3/4)^{u-1}$. Therefore,

$$|\mu| \leqslant E\left(|Y_1|\right) = \sum_{i=1}^{\infty} P\left(|Y_1| \geqslant i\right) \leqslant k + k\sum_{u=1}^{\infty}(3/4)^{u-1} = 5k.$$

Suppose $|\mu| \geqslant \frac{1}{10}$. By Wald's equation, we have

$$|\mu|E(T) = \left|E(S_T)\right| \leqslant E\left(|S_T|\right).$$

Since $|\mu|$ is finite, and bounded below by a positive constant, it follows that $E(T)$ is $O(n)$.

Now we consider the case when $|\mu| < \frac{1}{10}$. Observe that

$$\sigma^2 \leqslant E\left(Y_1^2\right) \leqslant k^2 + \sum_{t=k+1}^{\infty}(2t-1)P\left(|Y_1| \geqslant t\right) \leqslant 41k^2.$$

Let X be the collection of all sequences which form an epoch. Let X_S (respectively X_N, X_H, X_G) consist of all sequences in X whose penultimate term is S (respectively N, H, G). For any sequence x in X_S, define its neighbors in X_N, X_H, X_G to be the sequences which agree with x everywhere except in the penultimate position. Note that a sequence belongs to X_S, X_N, X_H or X_G with probability $\frac{1}{4}$. Clearly, a sequence in X_S and its neighbor in X_N cannot both have zero payoff. Therefore, one of them must contribute at least one unit towards $E(Y_1^2)$. Thus, $E(Y_1^2) \geqslant \frac{1}{4}$. Therefore, $\sigma^2 = E(Y_1^2) - \mu^2 > \frac{6}{25}$.

By Wald's equation, we have $E(S_T^2) = \sigma^2 E(T) + \mu^2 E(T^2) \geqslant \sigma^2 E(T)$. Since σ^2 is finite, and bounded below by a positive constant, it follows that $E(T)$ is $O(n^2)$. $\square$

Let $T_{s,k} \doteq \lfloor \frac{s}{30k} \rfloor$. Note that for $s \geqslant 75$, we have,

$$T_{s,k} < \left(\frac{0.76}{0.75}\right)^s.$$

Lemma 2. *Let $M_{s,k}$ denote the number of k-player hyperslowdel games lasting exactly s rounds, with fewer than $T_{s,k}$ epochs. Then,*

$$M_{s,k} < 4^{(k-0.1)s} \quad \textit{for } s \geqslant 75.$$

Proof. Observe that

$$M_{s,k} \leqslant 4^{s(k-1)} \sum_{r=0}^{T_{s,k}-1} \binom{s}{r} 3^{s-r} \leqslant 4^{s(k-1)} T_{s,k} 3^s \binom{s}{T_{s,k}}.$$

Since $\binom{m}{r} \leqslant (\frac{me}{r})^r$, we get,

$$M_{s,k} < 4^{ks} T_{s,k} (0.75)^s (60ek)^{s/30k} < 4^{ks} (0.76)^s (60ek)^{s/30k}.$$

Since $(60ek)^{1/30k} < \frac{4^{-0.1}}{0.76}$ for $k \geqslant 2$, we get $M_{s,k} < 4^{(k-0.1)s}$. $\square$

Lemma 3. *For $n \geqslant 80k^3$ and $s \geqslant 1200k^2 n$, there exist more than $4^{(k-0.1)s}$ hyperslowdel games lasting exactly s rounds with at least $T_{s,k}$ epochs.*

Proof. We construct the required number of hyperslowdel games lasting exactly s rounds and have at least $T_{s,k}$ epochs. Our games evolve in phases.

The first phase is restorative (hyperslowdel can start from any configuration) and ends when there are k tokens in the pot, the difference between the number of tokens in the possession of any pair of players is at most one, and it is the first player's turn to spin. This is accomplished as follows:

We begin with a sequence of Halbs, until there are only two tokens left in the pot. If there was only one token to begin with, we have a Shtel instead. We then have a (possibly empty) sequence of Nishts, until it is the first player's turn to spin. In every subsequent round, a player with the highest number of tokens gets Shtel, a player with the lowest number of tokens gets Halb, and everyone else gets Nishts. If at the end of any round the difference between the highest and lowest is at most one, we have $k-2$ rounds comprising Shtel for one of the (current) leaders and Nishts for everyone else, thus increasing the pot size from 2 to k. It is easy to see that the number of spins in the restorative phase is less than $2k^2 n$. Let m be the lowest number of tokens with any player at the end of this phase. Note that each player has $m + \delta$ tokens, with $m \geqslant n - 1$ and $\delta \in \{0, 1\}$.

In the second phase, we have $T_{s,k}$ rounds in which each player gets Ganz. This ensures that all the games we construct have at least $T_{s,k}$ epochs. The number of spins so far is less than $\frac{s}{25}$.

The third phase is divided into *gamelets*. A gamelet of length ℓ is a segment of ℓ spins. Note that all gamelets of length up to n which start from the initial configuration of dreidel are legal, since the payoff never drops below $-n$ or goes above n for such gamelets.

Let p be the unique integer such that $n - k - k^2 \leqslant pk^2 < n - k$ and let X be the collection of all gamelets of length $pk + 1$ that end with a Ganz. Let $g \in X$ and define $\rho(g) = (u_1, \ldots, u_{k-1})$ if and only if g gives payoffs $u_1, \ldots, u_{k-1}, -(u_1 + \cdots + u_{k-1})$

for players $P_1, \ldots, P_{k-1}, P_k$, respectively. Let $x_{u_1,\ldots,u_{k-1}}$ denote the number of gamelets g in X with $\rho(g) = (u_1, \ldots, u_{k-1})$. Observe that the payoffs are at least $-(pk+1)$ and at most $(k-1)(2p+1)$. Therefore,

$$\sum_{-pk-1 \leqslant u_1,\ldots,u_{k-1} \leqslant (k-1)(2p+1)} x_{u_1,\ldots,u_{k-1}} = 4^{pk}.$$

Note that if $g_1, g_2, \ldots, g_k$ are gamelets in X with $\rho(g_1) = \cdots \rho(g_k)$, then the concatenated gamelet $g_1 g_2 \ldots g_k$ gives zero payoff for every player. By Minkowski's inequality, there are at least

$$\sum_{-pk-1 \leqslant u_1,\ldots,u_{k-1} \leqslant (k-1)(2p+1)} x^k_{u_1,\ldots,u_{k-1}} \geqslant \frac{(4^{pk})^k}{(3pk)^{k-1}} > \frac{(\frac{k}{3})^{k-1}}{4^{k^2+k}} \frac{4^n}{n^{k-1}}$$

gamelets of length $pk^2 + k < n$ which give zero payoff for every player. Observe that for $n \geqslant 80k^3$, this number exceeds $4^{n(1-(1/20k))}$.

The third phase proceeds in a series of such concatenated gamelets with payoff zero until the next gamelet would increase the number of spins beyond $k(s-m-2)$. When this happens, we have (at most pk) rounds of Nishts till the number of spins is exactly $k(s-m-2)$.

This gives rise to at least

$$\begin{aligned} 4^{\frac{n[1-(1/20k)][k(s-m-2)-n-(s/25)]}{pk^2+k}} &> 4^{\frac{n[1-(1/20k)][ks-(s/600)-(s/25)]}{n}} \\ &= 4^{s(k-1/24)(1-1/20k)} > 4^{s(k-0.1)} \end{aligned}$$

different games.

The fourth and final phase has $m+2$ rounds. In the first m rounds, everyone gets Shtel. In the next round, everyone who has a token gets Shtel, and everyone else gets Nisht. In the last round, everyone gets Nisht, except P_k who gets Ganz, wins, and goes home.

Observe that we have constructed more than $4^{(k-0.1)s}$ different games lasting exactly ks spins, and with at least $T_{s,k}$ epochs for all $s \geqslant s_0$, where s_0 is sufficiently large. □

Let $p_{s,k}$ denote the probability that P_k goes home after exactly s rounds, and let $E_{s,k}$ denote the expected number of epochs in a game lasting exactly s rounds before P_k goes home.

Lemma 4. *For $n \geqslant 80k^3$ and $s \geqslant 1200k^2 n$, we have $E_{s,k} \geqslant \frac{s}{120k}$.*

Proof. From Lemma 3, there exist more than $4^{(k-0.1)s}$ different games lasting exactly ks spins, and with at least $T_{s,k}$ epochs for all $s \geqslant 1200k^2 n$. By Lemma 2, this exceeds the number of possible games with less than $T_{s,k}$ epochs. Thus, for all $s \geqslant 1200k^2 n$, we have $E_{s,k} \geqslant \frac{T_{s,k}}{2} \geqslant \frac{s}{120k}$. □

Theorem. *The expected number of spins in a game of dreidel between k players starting with n tokens each is $O(n^2)$, where the implied constant depends only on k.*

Proof. Let $p_{s,k}$ denote the probability that P_k goes home after exactly s rounds, and let $E_{s,k}$ denote the expected number of epochs in a game lasting exactly s rounds before P_k goes home. Let T denote the number of epochs in a game of hyperslowdel between k players starting with n tokens each. We have, $E(T) = \sum_{s=1}^{\infty} p_{s,k} E_{s,k}$.

Let U denote the number of spins in a game of hyperslowdel before P_k goes home. From Lemma 2, we get,

$$\begin{aligned} E(U) = \sum_{s=1}^{\infty} p_{s,k} ks &< 1200k^3 n + \sum_{s>1200k^2 n} p_{s,k} ks \\ &< 1200k^3 n + 120k^2 \sum_{s>1200k^2 n} p_{s,k} E_{s,k} \\ &< 1200k^3 n + 120k^2 E(T). \end{aligned}$$

It now follows from Lemma 1 that the expected number of spins before P_k goes home is $O(n^2)$, irrespective of the starting configuration. If P_k wins, the game is over. Otherwise, we have a game of hyperslowdel between at most $k-1$ players. Repeating the above argument, it is easy to see that the expected number of epochs in a game of hyperslowdel is $O(n^2)$. It follows that the expected number of spins in a game of dreidel between k players is $O(n^2)$, with the implied constant depending only on k. □

4. Variants

House rules vary, of course, from house to house. In the interests of expediting a game not without its share of lulls, it could be stipulated [5] that the spinner should double the pot upon Shtel, rather than donate just one token. Numerical evidence [4] suggests that this variant lasts $O(n^{1.389})$ spins, on average. A partial demystification is outlined below.

Define $\ell_n = 1 + \lceil \log_2(kn) \rceil$. If we ever have a sequence SS…SG of length ℓ_n, the game is over. Although the probability that a random sequence of length ℓ_n is of this type is $4^{-\ell_n}$, occurrences of N can be safely ignored, yielding an upper bound of $O(n^{1.585})$ on the expected number of spins.

Let $h_{\alpha,n} \doteq \lceil \alpha \log_2(kn) \rceil$ and $s_{\alpha,n} \doteq \lceil (1+\alpha) \log_2(kn) \rceil$. Consider all strings of length $1 + h_{\alpha,n} + s_{\alpha,n}$ with $h_{\alpha,n}$ occurrences of H, $s_{\alpha,n}$ occurrences of S, and ending with G. We assume a pre-processing step where all occurrences of N are removed. The optimum value of α is given by the solution to the quadratic equation $(1+2x)^2 = 9x(1+x)$, i.e., $\alpha = \frac{\sqrt{45}-5}{10} \approx 0.1708$. Accordingly, we get an upper bound of $O(n^{1.389})$ spins.

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